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Control of individual phase relationship between coupled oscillators using multilinear feedback

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Due to various technological and medical demands, several methods for controlling the dynamical behavior of coupled oscillators have been developed. In the present study, we develop a method to control the individual phase relationship between coupled oscillators, in which multilinear feedback is used to modify the interaction between the oscillators. By carrying out a simulation, we show that the phase relationship can be well controlled by using the proposed method and the control is particularly robust when the target coupling function is selected properly.

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I. INTRODUCTION

The synchronization of mutually interacting elements exhibiting regular rhythms is a well-known phenomenon observed in nature and has been extensively studied from physical, chemical, biological, and medical viewpoints [1–6]. Various types of phase relationships are found in these systems, such as the in-phase state [5,6], cluster state [6,7], and splay state [8]. Recently, techniques to control the phase relationship between such coupled oscillators have been stimulated. For example, electrical stimulation techniques for several neural diseases have been developed so that the stimulation effectively destroys the coherency of pathological neurons exhibiting in-phase synchronization [6,9–12]. Other examples include the realization of the locomotion of modular robots by introducing phase differences between modules that exhibit regular rhythms [13] and the adjustment of the phase difference between successive traffic signals through their local interaction in order to reduce traffic jams [14].

Several techniques have been developed to control the dynamical behavior of coupled oscillators. Rosenblum and Pikovsky proposed a linear feedback method and showed that the coherency of oscillators increases or decreases depending on the amplitude and delay of the feedback signal [12]. On the other hand, a method to control the functional form of the coupling function in a phase model has been proposed; in this method, various dynamical states such as the slow-switching state and cluster state can be obtained through nonlinear feedback [15–17]. Recently, we have proposed a tractable method to control the coupling function using multilinear feedback [18–20]. In this method, the sum of the signals generated from several oscillators is observed and the feedback signals proportional to the observable are applied to the system with multiple time delays. Since this method does not require individual signals from each oscillator to be measured and requires only several macroscopic observations to determine feedback parameters, it can be applied to various systems without practical restrictions.

Previous studies mainly focused on the control of the macroscopic dynamics of the overall system and, thus, the phase relationship between individual oscillators is still essentially uncontrollable. Indeed, although Kori *et al.* noted that the nonlinear feedback method can be applied to individual interaction between oscillators, they did not describe how the target coupling function was determined in order to achieve the desired phase relationship [16]. Recently, with the technological developments, there has been an increase in the demand for controlling not only the collective behavior of several oscillators but also the behavior of individual oscillators. Thus, it is urgently needed to develop a new theory to control the phase relationship between individual oscillators.

In the present paper, we aim to derive a method to control the phase relationship between individual oscillators, where the multilinear feedback reported previously [18–20] is employed to individual interactions between the oscillators. We will show by a simulation that the phase relationship can be well controlled using the present method and the control is particularly robust against noise and natural frequency distribution when the target coupling function is properly selected.

II. THEORY

We consider a system in which N oscillators are coupled to each other. The dynamics of each oscillator is described by the following equation:

$$\dot{\mathbf{x}}_i = \mathbf{F}(\mathbf{x}_i) + \epsilon_d \mathbf{f}_i(\mathbf{x}_i) + \sum_{j=1, j\neq i}^{N} \epsilon_{ij} \mathbf{P}_\mathbf{c}(\mathbf{x}_i, \mathbf{x}_j), \quad (1)$$

where $\mathbf{F}(\mathbf{x}_i)+\epsilon_d \mathbf{f}_i(\mathbf{x}_i)$ denotes a set of functions describing a limit cycle, with $\mathbf{F}(\mathbf{x}_i)$ a common part and $\epsilon_d \mathbf{f}_i(\mathbf{x}_i)$ the deviation from it for the ith oscillator. $\mathbf{P}_\mathbf{c}(\mathbf{x}_i,\mathbf{x}_j)$ is a function characterizing the way of coupling between the ith and jth oscillators and ϵ_{ij} is the coupling strength. We assume that ϵ_d and $\Sigma_{j=1,j\neq i}^{N}\epsilon_{ij}$ are sufficiently smaller than unity and that $\mathbf{F}(\mathbf{x}_i)$, $\mathbf{f}_i(\mathbf{x}_i)$, and $\mathbf{P}_\mathbf{c}(\mathbf{x}_i,\mathbf{x}_j)$ are the functions of $O(1)$. Then, Eq. (1) is reduced to a phase model as follows:

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$$\dot{\phi}_i = \bar{\omega} + \epsilon_d \omega_i + \sum_{j=1, j\neq i}^{N} \epsilon_{ij} q_c(\phi_i - \phi_j), \tag{2}$$

where $\omega_i = (1/2\pi)\int_0^{2\pi} d\theta \mathbf{Z}(\phi_i+\theta)\cdot \mathbf{f}_i[\mathbf{x}_0(\phi_i+\theta)]$ and $q_c(\phi_i - \phi_j) = (1/2\pi)\int_0^{2\pi} d\theta \mathbf{Z}(\phi_i+\theta)\cdot \mathbf{P_c}[\mathbf{x}_0(\phi_i+\theta), \mathbf{x}_0(\phi_j+\theta)]$, the latter of which is called coupling function. Here, we have defined $\mathbf{x}_0(\phi)$ as a point on the limit cycle at a phase ϕ and $\mathbf{Z}(\phi_i) \equiv \mathrm{grad}_{\mathbf{x}}\, \phi|_{\mathbf{x}=\mathbf{x}_0(\phi_i)}$.

We set the target phase relationship between each oscillator pair i and j as $\phi_i - \phi_j = \alpha_{ij}$. Note that $\alpha_{ij} = \alpha_{ih} + \alpha_{hj}$ is automatically satisfied for $h=1,2,\ldots$ and N. Let $p[\mathbf{x}_j(t)]$ be defined as the signal obtained from the jth oscillator, which is a single-valued function of $\mathbf{x}_j(t)$. Here, we assume that $p[\mathbf{x}_j(t)]$ can be measured independently for all j, which is in contrast to our previous studies [18–20] where only the sum of the signals from several oscillators was measurable. Then, we apply multilinear feedback signals independently to each oscillator as follows:

$$\dot{\mathbf{x}}_i = \mathbf{F}(\mathbf{x}_i) + \epsilon_d \mathbf{f}_i(\mathbf{x}_i) + \sum_{j=1, j\neq i}^{N} \epsilon_{ij} \mathbf{P_c}(\mathbf{x}_i, \mathbf{x}_j) + \sum_{j=1, j\neq i}^{N} \epsilon'_{ij} \sum_{m=1}^{2M+1} \Gamma_m^{(ij)} p_j[\mathbf{x}_j(t - \tau_m^{(ij)})]\mathbf{r}, \tag{3}$$

where $\mathbf{r}$ is a unit vector, which can be selected in an arbitrary manner, and ϵ'_{ij} is the strength of the feedback from the jth to ith oscillator. Although ϵ'_{ij} can be arbitrarily selected, it should be comparable to or larger than ϵ_{ij}, but should not be too large to validate the phase description. $\tau_m^{(ij)}$ and $\Gamma_m^{(ij)}$ are the time delay and strength of the mth feedback signal from the jth to ith oscillator, which we will specify below. The number of feedback signals is set at $2M+1$, where the definition of M will be described later. Equation (3) is reduced to the phase model as

$$\dot{\phi}_i = \bar{\omega} + \epsilon_d \omega_i + \sum_{j=1, j\neq i}^{N} \epsilon_{ij} q_c[\phi_i(t) - \phi_j(t)] + \sum_{j=1, j\neq i}^{N} \epsilon'_{ij} \sum_{m=1}^{2M+1} \Gamma_m^{(ij)} q_f\{\phi_i(t) - \phi_j[t - \tau_m^{(ij)}]\}, \tag{4}$$

where $q_f(\phi_i - \phi_j) = (1/2\pi)\int_0^{2\pi} d\theta \mathbf{Z}(\phi_i+\theta) p[\mathbf{x}_0(\phi_j+\theta)]\mathbf{r}$. The method for specifying the coupling functions $q_c(\phi_i - \phi_j)$ and $q_f(\phi_i - \phi_j)$ in actual systems is described elsewhere [21,22].

Suppose that Eq. (4) is equivalent to the following equation:

$$\dot{\phi}_i = \bar{\omega} + \epsilon_d \omega_i + \sum_{j=1, j\neq i}^{N} \epsilon'_{ij} \tilde{q}_{ij}[\phi_i(t) - \phi_j(t)], \tag{5}$$

where $\tilde{q}_{ij}[\phi_i(t) - \phi_j(t)]$ is the target coupling function, which is determined to obtain the target phase relationship and is derived in the following way. First, as a simple case, we consider the case of $N=2$. From Eq. (5), we obtain the following relation:

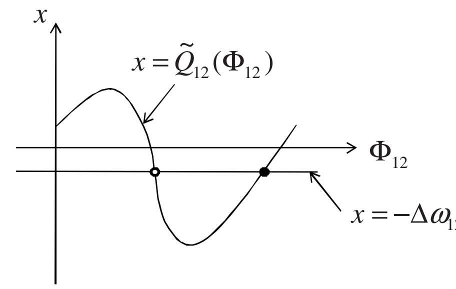

FIG. 1. Solutions of Eq. (6) given by intersections of $x = -\Delta\omega_{12}$ and $x = \tilde{Q}_{12}(\Phi_{12})$. Empty and filled circles denote stable and unstable solutions, respectively.

$$\dot{\Phi}_{12} = \Delta\omega_{12} + \tilde{Q}_{12}(\Phi_{12}), \tag{6}$$

where $\Phi_{12} = \phi_1 - \phi_2$, $\Delta\omega_{12} = \epsilon_d(\omega_1 - \omega_2)$, and $\tilde{Q}_{12}(\Phi_{12}) = \epsilon'_{12}\tilde{q}_{12}(\Phi_{12}) - \epsilon'_{21}\tilde{q}_{21}(-\Phi_{12})$. The steady solution of Eq. (6) is given by the intersection of the functions $x = -\Delta\omega_{12}$ and $x = \tilde{Q}_{12}(\Phi_{12})$ and the solution Φ_{12}^0 is stable if $\tilde{Q}'_{12}(\Phi_{12}^0) < 0$ and unstable if $\tilde{Q}'_{12}(\Phi_{12}^0) > 0$ (see Fig. 1). Thus, we have to select $\tilde{q}_{12}(\Phi_{12})$ and $\tilde{q}_{21}(-\Phi_{12})$ such that only $\Phi_{12} = \alpha_{12}$ becomes a unique stable solution of Eq. (6). In the case of $N \geq 3$, $\tilde{q}_{ij}(\phi_i - \phi_j)$ can be specified in the same manner as in the case of $N=2$. Namely, for each pair i and j, we select $\tilde{q}_{ij}(\Phi_{ij})$ and $\tilde{q}_{ji}(-\Phi_{ij})$ such that the two curves $x = -\Delta\omega_{ij}$ and $x = \tilde{Q}_{ij}(\Phi_{ij})$ uniquely intersect at a point $\Phi_{ij} = \alpha_{ij}$ with $\tilde{Q}'_{ij}(\Phi_{ij}) < 0$, where $\Phi_{ij} = \phi_i - \phi_j$, $\Delta\omega_{ij} = \epsilon_d(\omega_i - \omega_j)$, and $\tilde{Q}_{ij}(\Phi_{ij}) = \epsilon'_{ij}\tilde{q}_{ij}(\Phi_{ij}) - \epsilon'_{ji}\tilde{q}_{ji}(-\Phi_{ij})$.

Next, we determine the values of $\tau_m^{(ij)}$ and $\Gamma_m^{(ij)}$ by comparing each Fourier coefficient of Eqs. (4) and (5). Let the coupling functions be expanded to the Fourier series as $q_c(\Phi_{ij}) = \Sigma_k a_k^{(c)} \exp[ik\Phi_{ij}]$, $q_f(\Phi_{ij}) = \Sigma_k a_k^{(f)} \exp[ik\Phi_{ij}]$, and $\tilde{q}_{ij}(\Phi_{ij}) = \Sigma_{k=-M}^{M} \tilde{a}_k^{(ij)} \exp[ik\Phi_{ij}]$, where $a_{-k}^{(c)} = a_k^{(c)*}$, $a_{-k}^{(f)} = a_k^{(f)*}$, and $\tilde{a}_{-k}^{(ij)} = \tilde{a}_k^{(ij)*}$. M is defined as the highest harmonic of $\tilde{q}_{ij}(\Phi_{ij})$ since we aim to control the coupled oscillators with a finite number of such harmonics. Then, when $\tau_m^{(ij)}$ is set such that $\tau_m^{(ij)} \lesssim 2\pi/\bar{\omega}$ is satisfied, we can use the approximation $\phi_j(t - \tau_m^{(ij)}) \approx \phi_j(t) - \bar{\omega}\tau_m^{(ij)}$ and hence, we obtain the following relation up to the Mth harmonic:

$$\tilde{a}_k^{(ij)} = \frac{\epsilon_{ij}}{\epsilon'_{ij}} a_k^{(c)} + \sum_{m=1}^{2M+1} \Gamma_m^{(ij)} a_k^{(f)} \exp(ik\bar{\omega}\tau_m^{(ij)}). \tag{7}$$

The contribution of harmonics higher than M in $q_c(\Phi_{ij})$ and $q_f(\Phi_{ij})$ can be minimized by taking M sufficiently larger than the number of harmonics, within which $q_c(\Phi_{ij})$ and $q_f(\Phi_{ij})$ have non-negligible Fourier components. Equation (7) is rewritten in a matrix form as

$$\begin{pmatrix} A_0^{(ij)} \\ A_1^{(ij)} \\ A_2^{(ij)} \\ \vdots \\ A_M^{(ij)} \\ B_1^{(ij)} \\ B_2^{(ij)} \\ \vdots \\ B_M^{(ij)} \end{pmatrix} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \cos(\bar{\omega}\tau_1^{(ij)}) & \cos(\bar{\omega}\tau_2^{(ij)}) & \cdots & \cos(\bar{\omega}\tau_{2M+1}^{(ij)}) \\ \cos(2\bar{\omega}\tau_1^{(ij)}) & \cos(2\bar{\omega}\tau_2^{(ij)}) & \cdots & \cos(2\bar{\omega}\tau_{2M+1}^{(ij)}) \\ \vdots & \vdots & \ddots & \vdots \\ \cos(M\bar{\omega}\tau_1^{(ij)}) & \cos(M\bar{\omega}\tau_2^{(ij)}) & \cdots & \cos(M\bar{\omega}\tau_{2M+1}^{(ij)}) \\ \sin(\bar{\omega}\tau_1^{(ij)}) & \sin(\bar{\omega}\tau_2^{(ij)}) & \cdots & \sin(\bar{\omega}\tau_{2M+1}^{(ij)}) \\ \sin(2\bar{\omega}\tau_1^{(ij)}) & \sin(2\bar{\omega}\tau_2^{(ij)}) & \cdots & \sin(2\bar{\omega}\tau_{2M+1}^{(ij)}) \\ \vdots & \vdots & \ddots & \vdots \\ \sin(M\bar{\omega}\tau_1^{(ij)}) & \sin(M\bar{\omega}\tau_2^{(ij)}) & \cdots & \sin(M\bar{\omega}\tau_{2M+1}^{(ij)}) \end{pmatrix} \begin{pmatrix} \Gamma_1^{(ij)} \\ \Gamma_2^{(ij)} \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \Gamma_{2M}^{(ij)} \\ \Gamma_{2M+1}^{(ij)} \end{pmatrix}, \quad (8)$$

where $A_k^{(ij)}=\mathrm{Re}[\{\tilde{a}_k^{(ij)}-(\epsilon_{ij}/\epsilon'_{ij})a_k^{(c)}\}/a_k^{(f)}]$ and $B_k^{(ij)}=\mathrm{Im}[\{\tilde{a}_k^{(ij)}-(\epsilon_{ij}/\epsilon'_{ij})a_k^{(c)}\}/a_k^{(f)}]$. Thus, when the values of $\tau_1^{(ij)}$, $\tau_2^{(ij)}$, …, and $\tau_{2M+1}^{(ij)}$ are determined, the corresponding values of $\Gamma_1^{(ij)}$, $\Gamma_2^{(ij)}$, …, and $\Gamma_{2M+1}^{(ij)}$ can be derived by solving Eq. (8). Although there is no specific method for selecting $\tau_1^{(ij)}$, $\tau_2^{(ij)}$, …, and $\tau_{2M+1}^{(ij)}$ as far as they are comparable to or smaller than $2\pi/\bar{\omega}$, we should select them such that $\Sigma_{m=1}^{2M+1}|\Gamma_m^{(ij)}|$ is not large; otherwise the phase model will not be valid [18]. An easy method of selecting them is described in [18]. Here, since $\Sigma_{m=1}^{2M+1}|\Gamma_m^{(ij)}|$ cannot be smaller than $\mathrm{Max}[|A_k^{(ij)}|,|B_k^{(ij)}|]$, $\tilde{q}_{ij}[\phi_i(t)-\phi_j(t)]$ should be determined so that $\mathrm{Max}[|A_k^{(ij)}|,|B_k^{(ij)}|]$ is not large.

Although $\tau_m^{(ij)}$ and $\Gamma_m^{(ij)}$ should be calculated for each pair i and j in general cases, it is quite easy to derive $\tau_m^{(ij)}$ and $\Gamma_m^{(ij)}$ when the natural coupling is absent. In this case, once $\tilde{q}_{ij}(\psi)$ is determined for a certain pair of i and j, we can set $\tilde{q}_{i'j'}(\psi)$ for other pairs of i' and j' as $\tilde{q}_{i'j'}(\psi)=\tilde{q}_{ij}(\psi-\alpha_{ij}+\alpha_{i'j'})$. Thus, since $\tilde{a}_k^{(i'j')}$ is given as $\tilde{a}_k^{(i'j')}=\tilde{a}_k^{(ij)}\exp[ik(\alpha_{i'j'}-\alpha_{ij})]$, Eq. (7) is written in the form

$$\tilde{a}_k^{(i'j')} = \sum_{m=1}^{2M+1} \Gamma_m^{(ij)} a_k^{(f)} \exp[ik(\bar{\omega}\tau_m^{(ij)} + \alpha_{i'j'} - \alpha_{ij})]. \quad (9)$$

Hence, once $\tau_m^{(ij)}$ and $\Gamma_m^{(ij)}$ are determined, $\tau_m^{(i'j')}$ and $\Gamma_m^{(i'j')}$ can be easily determined such that $\tau_m^{(i'j')}=\tau_m^{(ij)}+(\alpha_{i'j'}-\alpha_{ij})/\bar{\omega}$ and $\Gamma_m^{(i'j')}=\Gamma_m^{(ij)}$.

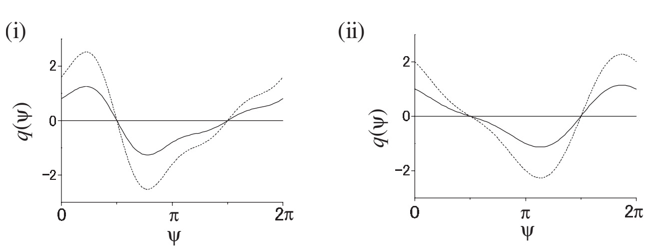

FIG. 2. Functional forms of $\tilde{q}_{ij}(\Phi_{ij})$ (solid line) and $\tilde{Q}(\Phi_{ij})/\epsilon'_{ij}$ (dashed line) employed in simulation. The two cases where (i) $\tilde{q}_{ij}(\Phi_{ij})=-\sin(\Phi_{ij}-\alpha_{ij})-0.4\sin 2(\Phi_{ij}-\alpha_{ij})-0.2\sin 3(\Phi_{ij}-\alpha_{ij})$ and (ii) $\tilde{q}_{ij}(\Phi_{ij})=-\sin(\Phi_{ij}-\alpha_{ij})+0.3\sin 2(\Phi_{ij}-\alpha_{ij})$, with $\alpha_{ij}=0.5\pi$, are shown.

III. SIMULATION

Let us confirm the validity of this method through a simulation. Here, we consider the case where Bonhoeffer–van der Pol oscillators, which exhibit typical limit-cycle oscillations [23], are interacting with each other. The model is given as follows:

$$\begin{pmatrix} 0.2\dot{u}_i \\ \dot{v}_i \end{pmatrix} = \begin{Bmatrix} -[1+c(i-1)]v_i + u_i - u_i^3/3 \\ u_i + 0.8 \end{Bmatrix} + \sum_{j=1,j\neq i}^{N} \epsilon_{ij}[u_j(t)-u_i(t)]\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sum_{j=1,j\neq i}^{N} \epsilon'_{ij} \sum_{m=1}^{2M+1} \Gamma_m^{(ij)} u_j[t-\tau_m^{(ij)}]\begin{pmatrix} 1 \\ 0 \end{pmatrix} + D\xi_i(t)\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad (10)$$

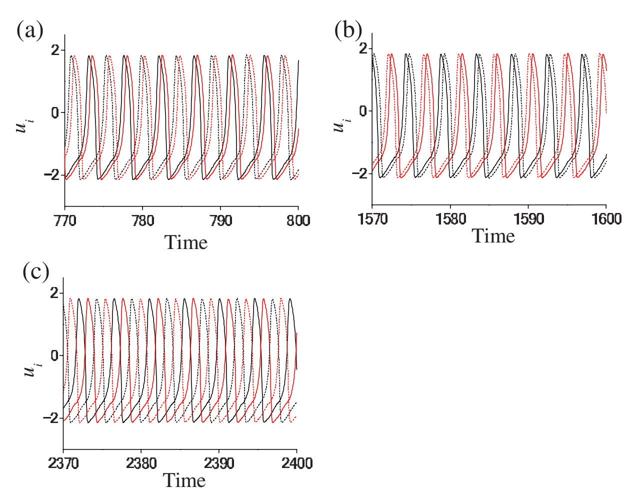

FIG. 3. (Color online) Waveform of each oscillator under the feedback for pattern (i) when $D=0.1$ and $c=0$. The waveforms of the first, second, third, and fourth oscillators are denoted by black solid , red (gray) solid, black dashed, and red (gray) dashed lines, respectively. The results for (a) $750\leq t\leq 800$, (b) $1550\leq t\leq 1600$, and (c) $2350\leq t\leq 2400$, where the target phase relationships $(\alpha_{12},\alpha_{13},\alpha_{14})$ are (a) $(0.2\pi,\pi,1.2\pi)$, (b) $(1.2\pi,0.2\pi,\pi)$, and (c) $(0.5\pi,\pi,1.5\pi)$, respectively, are shown.

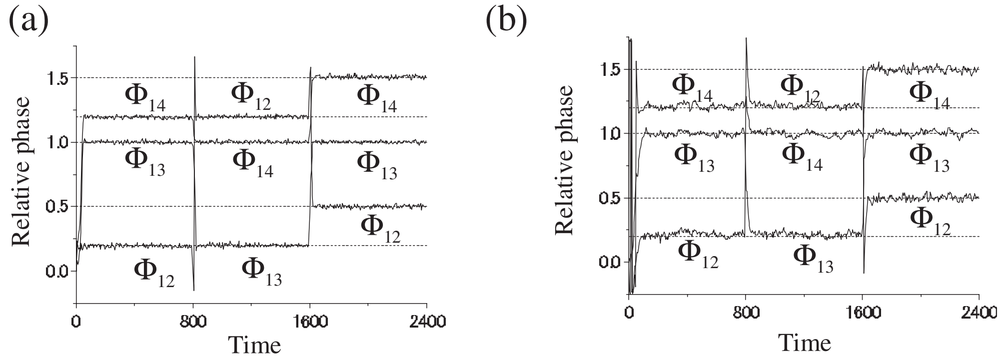

FIG. 4. Temporal evolutions of relative phases Φ_{12}, Φ_{13}, and Φ_{14} when $D=0.1$ and $c=0$: (a) pattern (i) and (b) pattern (ii). The target phase relationships $(\alpha_{12},\alpha_{13},\alpha_{14})$ are $(0.2\pi,\pi,1.2\pi)$, $(1.2\pi,0.2\pi,\pi)$, and $(0.5\pi,\pi,1.5\pi)$ for $0\leq t\leq 800$, $800<t\leq 1600$, and $1600<t\leq 2400$, respectively. The relative phases of 0.2π, 0.5π, π, 1.2π, and 1.5π are shown by dashed lines. Since the relative phase is 2π periodic, it is expressed within the range of $[-0.25\pi, 1.75\pi]$.

where the first, second, third, and fourth terms on the right-hand side denote the limit-cycle oscillation, natural coupling, feedback, and noise, respectively. The number of the oscillators, N, is taken as 4. c is a constant that is introduced so that the natural periods of the oscillators differ slightly from each other (see inset in Fig. 8). The natural coupling strength ϵ_{ij} is set at 0.01 for $(i,j)=(1,3)$, (3,1), (2,4), and (4,2), while it is set at 0.02 otherwise. The feedback strength ϵ'_{ij} is set at 0.05 for all pairs of i and j. The noise term is introduced to investigate the robustness of the control. $\xi_i(t)$ is random noise with a uniform distribution within the range of $[-1,1]$, which has a correlation neither between different time steps nor between different oscillators. D is the amplitude of noise. The initial condition is set at $u_i=v_i=1.5$ for all i. In the simulation, the Runge-Kutta method is employed with a time step of 0.02.

The functional forms of $q_c(\phi_i-\phi_j)$ and $q_f(\phi_i-\phi_j)$ are derived in the same manner as those in our previous study [see Figs. 1(a) and 1(b) in [18]]. Since $q_f(\phi_i-\phi_j)$ has non-negligible Fourier components up to around seventh harmonic, we have selected $M=8$. We set the target phase relationship as $(\alpha_{12},\alpha_{13},\alpha_{14})=(0.2\pi,\pi,1.2\pi)$, $(1.2\pi,0.2\pi,\pi)$, and $(0.5\pi,\pi,1.5\pi)$ for $0\leq t\leq 800$, $800<t\leq 1600$, and $1600<t\leq 2400$, respectively.

The functional form of $\tilde{q}_{ij}(\Phi_{ij})$ is determined for $c=0$ according to the method described in the previous section. Since the inclination of $\tilde{Q}_{ij}(\Phi_{ij})$ in the vicinity of $\Phi_{ij}=\alpha_{ij}$ is crucial for the stability of the target phase relationship, we have simulated two typical cases, (i) $q_{ij}(\Phi_{ij})=-\sin(\Phi_{ij}-\alpha_{ij})-0.4\sin 2(\Phi_{ij}-\alpha_{ij})-0.2\sin 3(\Phi_{ij}-\alpha_{ij})$ and (ii) $q_{ij}(\Phi_{ij})=-\sin(\Phi_{ij}-\alpha_{ij})+0.3\sin 2(\Phi_{ij}-\alpha_{ij})$, where $\tilde{Q}'_{ij}(\alpha_{ij})$ is large negative and small negative, respectively [the functional forms of $\tilde{q}_{ij}(\Phi_{ij})$ and $\tilde{Q}_{ij}(\Phi_{ij})/\epsilon'_{ij}$ for these two cases with $\alpha_{ij}=0.5\pi$ are shown in Fig. 2]. Hereafter, we call these two cases as "pattern (i)" and "pattern (ii)," respectively.

Figure 3 shows the waveform of each oscillator under the feedback for pattern (i) when $D=0.1$ and $c=0$. We find that the phase relationship changes as the target state changes. Let the relative phase Φ_{1j} $(j\geq 2)$ be defined as $\Phi_{1j}(t_1^{(K)})=2\pi(t_j^{(K')}-t_1^{(K)})/(t_1^{(K+1)}-t_1^{(K)})+2n\pi$, where n is an integer and $t_1^{(K)}$ and $t_j^{(K')}$ denote the time when the first and jth oscillators take the maximum value of u at the Kth and K'th cycles, respectively, with K and K' satisfying $t_1^{(K)}\leq t_j^{(K')}<t_1^{(K+1)}$. The temporal evolution of the relative phase is shown in Fig. 4(a). It is clear that the target phase relationship is obtained through the feedback, although it fluctuates slightly due to the noise. We have found that the phase relationship can be

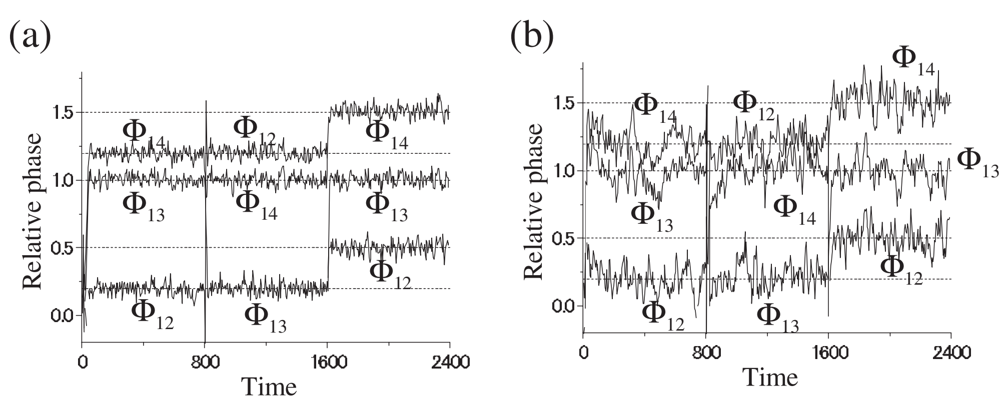

FIG. 5. Temporal evolutions of relative phases Φ_{12}, Φ_{13}, and Φ_{14} when $D=0.5$ and $c=0$: (a) pattern (i) and (b) pattern (ii). The target phase relationships $(\alpha_{12},\alpha_{13},\alpha_{14})$ are $(0.2\pi,\pi,1.2\pi)$, $(1.2\pi,0.2\pi,\pi)$, and $(0.5\pi,\pi,1.5\pi)$ for $0\leq t\leq 800$, $800<t\leq 1600$, and $1600<t\leq 2400$, respectively. The relative phases of 0.2π, 0.5π, π, 1.2π, and 1.5π are shown by dashed lines. Since the relative phase is 2π periodic, it is expressed within the range of $[-0.2\pi, 1.8\pi]$.

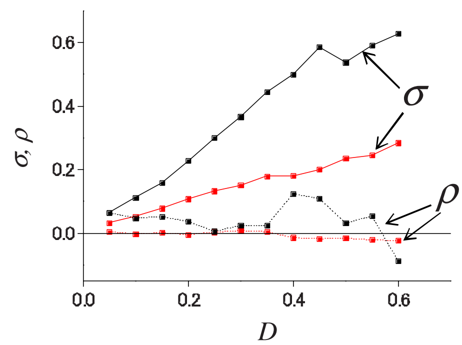

FIG. 6. (Color online) Values of σ (solid lines) and ρ (dashed lines) when D is varied with $c=0$. The results for patterns (i) (black lines) and (ii) (red (gray) lines) are shown.

well controlled even when the initial conditions of u_i and v_i are varied (data not shown). Figure 4(b) shows the result for pattern (ii). Although the target phase relationship is well reproduced in this case, the fluctuation of the relative phase is larger than that in the case of pattern (i).

Figure 5 shows the result when the noise amplitude D is increased. Although the target phase relationship is still maintained, the fluctuation increases, particularly for pattern (ii). To evaluate the deviation from the target phase relationship quantitatively, we have defined the parameters ρ and σ as $\rho=\sum_{l=1}^{3}\sum_{j=2}^{4}\langle\Phi_{1j}(t)-\alpha_{1j}(t)\rangle_l$ and $\sigma=\sum_{l=1}^{3}\sum_{j=2}^{4}\langle|\Phi_{1j}(t)-\langle\Phi_{1j}(t)\rangle_l|\rangle_l$, which characterize the average deviation and magnitude of the fluctuation, respectively. Here, $\langle\cdots\rangle_1$, $\langle\cdots\rangle_2$, and $\langle\cdots\rangle_3$ indicate the average over time within the ranges of $200\leq t\leq 800$, $1000\leq t\leq 1600$, and $1800\leq t\leq 2400$, respectively, where transient processes are not included in the evaluation. Figure 6 shows the values of σ and ρ when D is varied for patterns (i) and (ii). We find that σ increases almost linearly with D for both patterns, where the increase rate is higher for pattern (ii) than for pattern (i). On the other hand, the value of ρ is generally around 0. We notice that ρ hardly varies with D for pattern (i), while it varies to some extent for pattern (ii).

Figure 7 shows the result for $c=0.05$, where the natural periods of the fastest and slowest oscillators differ by $\sim$10% (see inset in Fig. 8). We find that the phase relationship generally deviates from the target one. In particular, the deviation is larger for pattern (ii) than for pattern (i). Figure 8 shows the relation between ρ and c for patterns (i) and (ii). Although ρ increases significantly with increasing c for pattern (ii), the increase rate is relatively low for pattern (i). Thus, the control is found to be relatively robust against noise and natural frequency distribution for pattern (i) as compared to pattern (ii).

IV. DISCUSSION

We have proposed a method to control the phase relationship between coupled oscillators. Although several methods have been developed to control the macroscopic dynamical behavior of coupled oscillators [12,15–20], the theoretical basis for controlling the phase relationship between individual oscillators has not been proposed thus far. In the present study, we have developed a theory, in which the multilinear feedback is employed for each interaction between the oscillators so that the individual phase relationship is controlled, and have confirmed its validity through a simulation.

We have found that the phase relationship can be well controlled regardless of the initial condition. This is thought to be because the system has unique basin of attraction at the target state. Actually, in the case of $N=2$, the target phase relationship $\phi_1-\phi_2=\alpha_{12}$ is a unique stable state, as shown by Eq. (6) (see Fig. 1). Even in the case of $N\geq 3$, the phase difference between the ith and jth oscillators will become α_{ij} for each pair i and j, and hence, the system will be finally attracted into the target phase relationship.

Through the simulation, we have shown that the two different patterns of $\tilde{q}(\Phi_{ij})$ have both led to the target phase relationship. However, the response to noise is quite different for these two patterns; the fluctuation of the relative phase of pattern (i) is smaller than that of pattern (ii). This difference is qualitatively explained in the following manner. First, let us consider the case of $N=2$. Since $\tilde{Q}'_{12}(\alpha_{12})$ is largely negative for pattern (i), the deviation from the target phase rela-

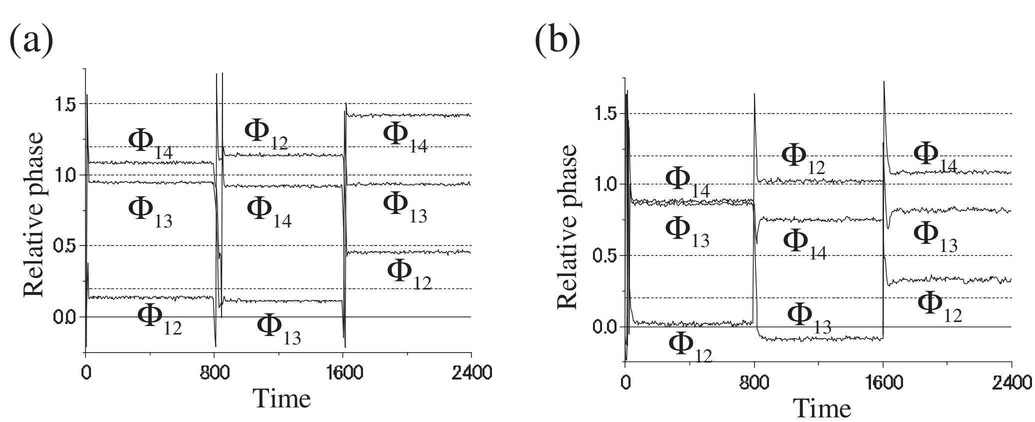

FIG. 7. Temporal evolutions of relative phases Φ_{12}, Φ_{13}, and Φ_{14} when $D=0.05$ and $c=0.05$: (a) pattern (i) and (b) pattern (ii). The target phase relationships $(\alpha_{12},\alpha_{13},\alpha_{14})$ are $(0.2\pi,\pi,1.2\pi)$, $(1.2\pi,0.2\pi,\pi)$, and $(0.5\pi,\pi,1.5\pi)$ for $0\leq t\leq 800$, $800< t\leq 1600$, and $1600< t\leq 2400$, respectively. The relative phases of 0.2π, 0.5π, π, 1.2π, and 1.5π are shown by dashed lines. Since the relative phase is 2π periodic, it is expressed within the range of $[-0.25\pi,1.75\pi]$.

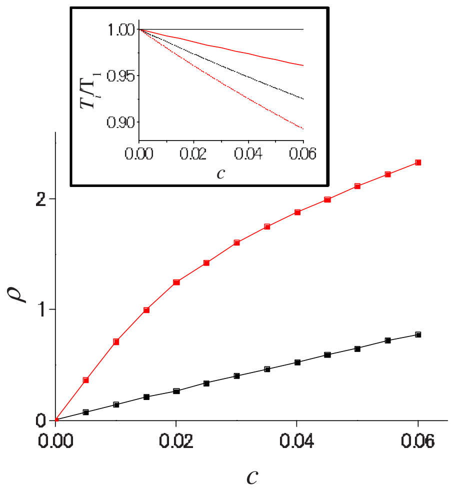

FIG. 8. (Color online) Relation between ρ and c when $D=0.05$. The results for patterns (i) (black lines) and (ii) (red (gray) lines) are shown. In the inset, the ratio of the natural period of the ith oscillator T_i to that of the first oscillator T_1 when c is varied is shown. The black solid, red (gray) solid, black dashed, and red (gray) lines denote the results for $i=1$, 2, 3, and 4, respectively.

tionship caused by a perturbation diminishes faster for pattern (i) than for pattern (ii), which is easily proved by linearizing Eq. (6) around $\Phi_{12}=\alpha_{12}$. Thus, the phase relationship does not deviate significantly from the target one for pattern (i) due to noise. In the case of $N\geq 3$, the same consideration will be applicable for each interaction between the ith and jth oscillators, and hence, the fluctuation will be relatively small for pattern (i) as a whole. This consideration can be applied regardless of the value of α_{ij}.

In the presence of the natural frequency distribution, the phase relationship deviates from the target one, where the deviation is larger for pattern (ii) than for pattern (i). This can be understood by considering the stable solution of Eq. (6) (see Fig. 1). As observed from Fig. 2, the value of Φ_{12} at the intersection of $x=-\Delta\omega_{12}$ and $x=\tilde{Q}_{12}(\Phi_{12})$, with $\tilde{Q}'_{12}(\Phi_{12})<0$, hardly changes with $\Delta\omega_{12}$ for pattern (i), whereas it changes considerably for pattern (ii). Since this argument holds for each oscillator pair i and j, the phase relationship under feedback is hardly affected by the change in the natural frequency for pattern (i).

Thus, for robust control, it is preferable to select $\tilde{q}_{ij}(\Phi_{ij})$ such that $\tilde{Q}'_{ij}(\alpha_{ij})$ is largely negative. However, $\tilde{q}_{ij}(\Phi_{ij})$ has to be selected such that $\max[|A_k^{(ij)}|,|B_k^{(ij)}|]$ is not large. Since A_k and B_k contain $a_k^{(f)}$ in their denominator, we can select $\tilde{q}_{ij}(\Phi_{ij})$ rather plausibly when $q_f(\phi_i-\phi_j)$ has non-negligible Fourier components for higher harmonics.

Since $\tau_m^{(ij)}$ and $\Gamma_m^{(ij)}$ have to be calculated for each pair i and j in general cases, the amount of calculation becomes extremely large as N increases. However, when the natural coupling is absent, such as in artificial systems, they can be derived in a simple manner as described in Sec. II. Moreover, in this case, we have only to apply feedback signals to adjacent oscillators but not to all oscillators because it is sufficient to tune the phase relationship between adjacent ones. Thus, in the case where the natural coupling is absent, the present method can be easily applied even when N is large.

The phase relationship can also be controlled by applying strong periodic signal to each oscillator with a delay time that corresponds to the target phase difference. Although this method is simple, it is not plausible for practical applications because such strong signal often damages the oscillators or tissues surrounding them, particularly when this method is applied to biological systems. In this context, the present method has an advantage that it can change the phase relationship by applying weak feedback signals and, thus, various practical applications are expected.

The present method is applicable only when the phase description is valid. Thus, it cannot be applied to the cases where properties of individual oscillators differ considerably from each other, the natural coupling between oscillators is not sufficiently weak, and the system has large noise. The control of the phase relationship in such cases is an open problem. Another problem is that the transient process leading to the target phase relationship cannot be controlled at the present stage. In actual systems, transient process has to be controlled, e.g., smooth transition to a different dynamical state in robotics. Thus, the theory has to be further developed in the future.

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